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Congruence subgroups, elliptic cohomology, and the Eichler–Shimura map

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Abstract

In 1986 Landweber [7] introduced the connective and periodic elliptic cohomology theories whose coefficient rings can be interpreted as a ring of modular functions for certain congruence subgroups of $SL_2(\mathbb{Z})$. One of the open questions in the subject has been to produce a geometric definition of these theories.

Nishida [8] defines a spectrum X_Γ based on the congruence subgroup Γ, which is related to the connective elliptic cohomology theory when $\Gamma = \Gamma_0(2)$. X_Γ has a stable summand X_{Γ^-}, and he proposes that the Eichler-Shimura map gives a real vector space isomorphism from the modular forms of Γ of weight $2k + 2$ to the real cohomology of X_{Γ^-} in dimension $4k + 1$ for $\Gamma = \Gamma_0(2)$. One of our main results is a proof of this claim when $k > 0$ for $\Gamma = \Gamma_0(p)$ and when $k \geq 0$ for $\Gamma = \Gamma_0(2)$ or $\Gamma = SL_2(\mathbb{Z})$. Using obstruction theory, we are able to construct a non-trivial geometric map from $\Sigma^3 X_{\Gamma^-}$ to the 3-connected cover of the spectrum representing the connective theory which is an equivalence through dimension 4. We also produce a stable splitting of X_Γ and of the spectrum representing the periodic theory introducd by Baker [2].

Keywords: Elliptic cohomology; Modular forms

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1. Introduction

Our primary result is that the Eichler–Shimura map can be restricted to an isomorphism from the homotopy of the spectrum representing Landweber's connective elliptic cohomology theory, Ell^c, to the cohomology of a stable summand X_{Γ^-} of the space X_Γ. We are able to construct a non-trivial geometric map from $\Sigma^3 X_{\Gamma^-}$ to the 3-connected cover of Ell^c which is an equivalence through dimension 4. Although I do not believe that this map is an equivalence in all dimensions, I hope that it may give us some geometric information about Ell^c. However, I do not understand the precise relationship at this time.

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In Section 2 we define congruence subgroups $\Gamma \subseteq SL_2(\mathbb{Z})$ and the space

$$X_\Gamma = E\Gamma \times_\Gamma (\mathbb{C}P^\infty)^{\times 2}.$$

We show that

$$H^{2k+1}(X_\Gamma; \mathbb{Z}(\tfrac{1}{6})) \cong H^1(\Gamma; (\mathbb{Z}(\tfrac{1}{6})[x, y])^{2k})$$

and that $H^*(X_\Gamma; \mathbb{Z}(\frac{1}{6}))$ is isolated in dimension 0 and $2k+1$ if $-Id \notin \Gamma$ and dimensions 0 and $4k+1$ if $-Id \in \Gamma$.

In Section 3 we define $M_k^R(\Gamma)$, the modular forms of weight k with coefficients in R, and the Eichler–Shimura map

$$\varphi : M_{k+2}^{\mathbb{C}}(\Gamma) \to H^1(\Gamma; W^{2k}),$$

where $W = \mathbb{R}[x, y]$. This is a $\mathbb{R}$-vector space map. As our results on the cohomology of X_Γ would lead us to suspect, $M_*^{\mathbb{C}}(\Gamma) = 0$ for all odd weights if $-Id \in \Gamma$. We also show that Shimura's action of Γ on W^{2k} is isomorphic to the polynomial replacement action we obtain from X_Γ.

In Section 4 we will use the Eichler–Shimura map to show that for certain congruence subgroups, X_Γ splits stably into two pieces, X_{Γ^-} and X_{Γ^+}, and we have an $\mathbb{R}$-module homomorphism

$$M_{2k+2}^{\mathbb{R}}(\Gamma) \to H^{4k+1}(X_{\Gamma^-}; \mathbb{R}) \quad \text{for } k \geq 0.$$

We will show that this map is an isomorphism for $k \geq 1$, for certain subgroups and is an isomorphism for $SL_2(\mathbb{Z})$ and $\Gamma_0(2)$ when $k = 0$. Nishida proposed this result in [8] for $\Gamma = \Gamma_0(2)$.

In Section 5 we define Landweber's connective elliptic cohomology theory Ell^c. The relevant point is that $Ell_{4k}^c \cong M_{2k}^{\mathbb{Z}(1/6)}(\Gamma_0(2))$. Our results from Section 4 show that $Ell_{4k}^c \otimes \mathbb{R} \cong H^{4k}(\Sigma^3 X_{\Gamma_0(2)^-}; \mathbb{R})$. We define a non-trivial geometric map from $\Sigma^3 X_{\Gamma_0(2)^-}$ to the 3-connected cover of Ell^c which is an equivalence through dimension 4. Using the standard splitting of $(\mathbb{C}P^\infty)^{\times 2}$, we are able to construct a dimensionwise stable splitting of X_Γ localized at p.

In Section 6, we construct a stable splitting of Baker's periodic theory similar to our splitting of X_Γ. The idempotents we form are constructed using the elliptic cohomology Adams operations that Baker introduces.

2. Congruence subgroups and the cohomology of X_Γ

Let $SL_2(\mathbb{Z})$ denote the group of 2×2 matrices with integer entries and determinant equal to 1. For any positive integer N define the *principal congruence subgroup* of level N by

$$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv Id \bmod N\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \,\middle|\, a \equiv d \equiv 1 \bmod N \text{ and } b \equiv c \equiv 0 \bmod N \right\}.$$

This subgroup is normal and has finite index in $SL_2(\mathbb{Z})$ since it is the kernel of the epimorphism

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N)$$

given by reduction mod N [10, p. 20]. We call any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ that contains $\Gamma(N)$ a *congruence subgroup* of level N. Of particular interest to us are the congruence subgroups of level N defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}.$$

It is well-known that $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated free product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ where $\mathbb{Z}/4$, $\mathbb{Z}/2$, and $\mathbb{Z}/6$ are generated by

$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_6 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

respectively [4, 9]. $\Gamma(N)$ is a free group for all $N > 2$ [3, p. 54], and $\Gamma(2)$ has only 2-torsion [3, p. 41].

Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. Then Γ acts on $\mathbb{Z} \times \mathbb{Z}$ by right multiplication, and thus on $T^2 = B(\mathbb{Z} \times \mathbb{Z})$ and on $(\mathbb{C}P^\infty)^{\times 2} = BT^2$. We follow Nishida [8] and define X_Γ by the Borel construction

$$X_\Gamma = E\Gamma \times_\Gamma (\mathbb{C}P^\infty)^{\times 2}.$$

Let $R = \mathbb{Z}(\frac{1}{6})$ and $U = H^*((\mathbb{C}P^\infty)^{\times 2}; R) = R[x, y]$ with $|x| = |y| = 2$.

The induced action of Γ on the left of $H^{2k}((\mathbb{C}P^\infty)^{\times 2}; R) = U^{2k}$ is given by polynomial replacement. That is, $U^{2k} = R^{k+1}$ is a free $(k+1)$-dimensional R-module with generators $x^k, x^{k-1}y, \ldots, xy^{k-1}$. Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and consider x as the column vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and y as the column vector $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Then α acts by left multiplication so that

$$\alpha \cdot x = ax + cy \quad \text{and} \quad \alpha \cdot y = bx + dy.$$

We can extend multiplicatively to U^{2k} by

$$\alpha \cdot \left(\sum_{j=0}^{k} c_j x^{k-j} y^j \right) = \sum_{j=0}^{k} c_j (ax + cy)^{k-j} (bx + dy)^j.$$

The following results will be useful in calculating the cohomology of X_Γ.

Lemma 1. *Let $SL_2(\mathbb{Z})$ act on U^{2k} by polynomial replacement, and let h, $l \in \mathbb{Z}$ be non-zero. Then*

$$(U^{2k})^{\gamma_0^h} = R\{y^k\} \quad \textit{and} \quad (U^{2k})^{\gamma_\infty^l} = R\{x^k\}.$$

Proof. First, notice that $\gamma_0^h = \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$ and

$$\gamma_0^h \cdot x^{k-r} y^r = (x + hy)^{k-r} y^r = x^{k-r} y^r + \sum_{j=1}^{k-r} \binom{k-r}{j} h^j x^{k-r-j} y^{j+r}.$$

If $m = \sum_{r=t}^{k} d_r x^{k-r} y^r \in U^{2k}$ where $d_t \neq 0$, then

$$\gamma_0^h \cdot m = m + (k-t) d_t h x^{k-r-1} y^{r+1} + \text{higher powers of } y.$$

Thus, the only fixed points occur when $t = k$, giving $(U^{2k})^{\gamma_0^h} = R\{y^k\}$. Since $\gamma_\infty^l = \left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$, a similar analysis gives $(U^{2k})^{\gamma_\infty^l} = R\{x^k\}$. □

Corollary 2. *If Γ is a subgroup of $SL_2(\mathbb{Z})$ of finite index, then $U^\Gamma = R$.*

Proof. Since $\gamma_\infty, \gamma_0 \in SL_2(\mathbb{Z})$ and Γ has finite index, some finite power of each must lie in Γ. Thus, Γ contains elements of the form γ_0^h and γ_∞^l. But Lemma 1, gives that $U\gamma_0^h = R[y]$ and $U\gamma_\infty^l = R[x]$. The only common fixed points are the coefficients R. □

Proposition 3. *Let $\Gamma \subseteq SL_2(\mathbb{Z})$ and $R = \mathbb{Z}(\frac{1}{6})$. Then for all $i > 1$ and for all Γ-modules M, $H^i(\Gamma; M) \otimes R = 0$. Thus, if $i > 1$, $H^i(\Gamma; M)$ has no free part and only 2 and 3 torsion.*

Proof. Shimura [10, p. 22] gives $[SL_2(\mathbb{Z}) : \Gamma(3)] = |SL_2\mathbb{Z}/3| = 3^3(1 - \frac{1}{3^2}) = 24$. Since $\Gamma(3)$ is a free group, $\Gamma(3)$ has no cohomology in dimension 2 or above. Further, restriction followed by the transfer is multiplication by the index of the subgroup, so that the composition

$$H^i(SL_2(\mathbb{Z}); M) \xrightarrow{res^*} H^i(\Gamma(3); M) \xrightarrow{tr^*} H^i(SL_2(\mathbb{Z}); M)$$

gives that the cohomology of $SL_2(\mathbb{Z})$ in dimension 2 and above is annihilated by 24. Thus, $H^i(SL_2(\mathbb{Z}); M) \otimes R = 0$ for all $i > 1$.

To calculate $H^i(\Gamma; M)$, we will apply Shapiro's Lemma [3, p. 73]. Recall that the coinduced module of M is defined by

$$Coind_\Gamma^{SL_2(\mathbb{Z})} M = Hom_{\mathbb{Z}\Gamma}(\mathbb{Z}SL_2(\mathbb{Z}), M).$$

Then Shapiro's Lemma states that

$$H^*(\Gamma; M) \cong H^*(SL_2(\mathbb{Z}); Coind_\Gamma^{SL_2(\mathbb{Z})} M).$$

Therefore, if $i > 1$,

$$0 = H^i(SL_2(\mathbb{Z}); Coind_\Gamma^{SL_2(\mathbb{Z})} M) \otimes R = H^i(\Gamma; M) \otimes R.$$

Thus, $H^*(\Gamma; M)$ has only 2 or 3 torsion in dimensions above 1. □

Proposition 4.

$$H^*(X_\Gamma; R) = \begin{cases} R & * = 0, \\ H^1(\Gamma; U^{2k}) & * = 2k+1, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Consider the fibration

$$(\mathbb{C}P^\infty)^{\times 2} \to X_\Gamma \to B\Gamma.$$

The associated Serre spectral sequence gives

$$E_2^{r,s} = H^r(B\Gamma; H^s((\mathbb{C}P^\infty)^{\times 2}; R)) = H^r(\Gamma; U^s).$$

Since $H^r(\Gamma; U^s) = 0$ for all $r > 1$ and $U^s = 0$ if s is odd, we have $E_2^{r,s} = 0$ unless s is even, and $r = 0$ or 1. Thus, there are no differentials, and the spectral sequence collapses. But $E_2^{0,s} = H^0(\Gamma; U^s) = (U^s)^\Gamma = 0$ for all $s > 0$ by Corollary 2. Thus, the E_2 term is isolated in $E_2^{0,0}$ and $E_2^{1,2k}$, and we have the desired result. □

Furthermore, if Γ is a congruence subgroup such that $-Id \in \Gamma$, then we can show that the cohomology of X_Γ is isolated in dimensions $4k+1$. Notice that $\Gamma_0(N)$ and $SL_2(\mathbb{Z})$ are such groups.

Proposition 5. *If $-Id \in \Gamma$, then $H^1(\Gamma; U^{2k}) = 0$ if k is odd.*

Proof. Let $\bar{\Gamma} = \Gamma/\{\pm Id\}$, so we have a short exact sequence

$$1 \to \{\pm Id\} \to \Gamma \to \bar{\Gamma} \to 1$$

with an assoicated Serre spectral sequence $E_2^{r,s} = H^r(\bar{\Gamma}; H^s(\{\pm Id\}; U^{2k}))$. Using the standard resolution for finite cyclic groups [3, p. 58], we get $H^s(\{\pm Id\}; U^{2k}) = 0$ for all s, so that the E_2-term is identically zero. □

Corollary 6. *If $-Id \in \Gamma$, then*

$$H^*(X_\Gamma; R) = \begin{cases} R & * = 0, \\ H^1(\Gamma; U^{2k}) & * = 4k+1, \\ 0 & \textit{otherwise}. \end{cases}$$

3. Modular forms and the Eichler–Shimura map

Let $\mathbb{C}$ denote the complex numbers and let $\mathscr{H} = \{x \in \mathbb{C} \mid Im(z) > 0\}$ be the upper complex half-plane. Define the action of $SL_2(\mathbb{Z})$ on $\mathscr{H}$ in the usual way by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Let $\mathbb{Q}$ denote the rational numbers and let $\mathscr{H}^* = \mathscr{H} \cup \mathbb{Q} \cup \infty$. Since $-Id$ acts trivially on $\mathscr{H}^*$, it is often more convenient to deal with $\overline{SL_2(\mathbb{Z})} = SL_2(\mathbb{Z})/\{\pm Id\}$. For any congruence subgroup Γ, define $\bar{\Gamma} = \Gamma/(\Gamma \cap \{\pm Id\})$. Note that if $-Id \in \Gamma$, then $\Gamma = \{\pm Id\} \times \bar{\Gamma}$, and if $-Id \notin \Gamma$, then $\Gamma = \bar{\Gamma}$.

For any $\Gamma \subseteq SL_2(\mathbb{Z})$, define the *cusps* of Γ to be the equivalence classes of $\mathbb{Q} \cup \infty$ under the Γ action. An element $\gamma \in \Gamma$ that fixes a cusp is called *parabolic*. For example,

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ fixes } \infty \quad \text{and} \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ fixes } 0.$$

The following results are easily verified.

Lemma 7. *$SL_2(\mathbb{Z})$ has a single cusp. We will usually choose ∞ as the representative of the cusp.*

Lemma 8. *If p is prime, then $\Gamma_0(p)$ has two cusps, 0 and ∞.*

We will follow the definition of modular forms given by Koblitz [5, p. 124]. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and let $f(z)$ be a function from $\mathscr{H}^*$ to $\mathbb{C} \cup \infty$. We will use the symbol $f|[\gamma]_k$ to denote the function whose value at z is $(cz + d)^{-k} f(\gamma \cdot z)$. That is,

$$f(z)|[\gamma]_k = (cz + d)^{-k} f(\gamma \cdot z).$$

Let $f(z)$ be a holomorphic function on $\mathscr{H}$, let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup of level N, and let $k \in \mathbb{Z}$. Then $f(z)$ is a *modular form* of weight k for Γ if

$$f|[\gamma]_k = f \quad \text{for all } \gamma \in \Gamma,$$

and if for any $\tau \in SL_2(\mathbb{Z})$,

$$f(z)|[\tau]_k \text{ has the form } \Sigma a_n q_N^n, \tag{1}$$

where $q_N = e^{(2\pi i z/N)}$ and $a_n = 0$ for $n < 0$. We call a modular form a *cusp form* if, in addition, $a_0 = 0$ for all $\tau \in SL_2(\mathbb{Z})$.

Notice that if $-Id \in \Gamma$, then there are no non-zero modular forms of odd weight. That is, $f|[-Id]_k = (-1)^{-k} f(z)$. And if k is odd, $f|[-Id_k] = f$ implies that $f = 0$.

The second condition (1) is actually a set of conditions, one for each cusp [5, p. 126]. For even weights, the condition is given as follows: Let s be a cusp of Γ, and $\rho \in SL_2(\mathbb{Z})$ such that $\rho \cdot \infty = s$. Then

$$\rho^{-1}\Gamma_s\rho \times \{\pm Id\} = \left\langle \pm\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^m \right\rangle$$

for some h, and (1) gives

$$f|[\rho]_{2k} = \sum b_j q_h^j \quad \text{where } q_h = e^{(2\pi i z/h)}.$$

In particular, at $s = \infty$, $\rho = Id$ and $\rho^{-1}\Gamma_\infty\rho = \Gamma_\infty$. Thus,

$$f|[\rho]_{2k} = f(z) = \sum a_j q_h^j \quad \text{for some } h. \tag{2}$$

At $s = 0$, $\rho = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$f|[\rho]_{2k} = z^{-2k} f\left(-\frac{1}{z}\right) = \sum b_j q_h^j \quad \text{for some } h. \tag{3}$$

We call the expansion of f at ∞ the *q-expression* of f. If $R \subseteq \mathbb{C}$ is some extension of $\mathbb{Z}$, we call the set of modular forms of Γ of weight k whose q-expansions have coefficients in R the *R modular forms* of Γ and denote them by $M_k^R(\Gamma)$. Similarly, we define the *R cusp forms* to be the cusp forms of Γ of weight k whose q-expansions have coefficients in R and denote them by $S_k^R(\Gamma)$.

Let W be the polynomial ring $\mathbb{R}[x, y]$ with $|x| = |y| = 2$, and let W^{2k} denote the homogeneous polynomials of degree k. The Eichler–Shimura map $\varphi : M_{k+2}^{\mathbb{C}}(\Gamma) \to H^1(\Gamma; W^{2k})$ is a map of $\mathbb{R}$-vector spaces that restricts to a $\mathbb{R}$-vector space isomorphism $S_{2k+2}^{\mathbb{C}}(\Gamma) \to H_p^1(\Gamma; W^{4k})$. The target is a subgroup of $H^1(\Gamma; W^{4k})$ that is defined below.

Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup, and let M be a Γ-module. Let P be the set of all parabolic elements of Γ. Define the *parabolic cocycles* by

$$Z_p^1(\Gamma; M) = \{u \in Z^1(\Gamma; M) \mid u(\pi) \in (\pi - 1)M \text{ for all } \pi \in P\}.$$

Define the *parabolic cohomology* of Γ with coefficients in M to be

$$H_p^1(\Gamma; M) = \frac{Z_p^1(\Gamma; M)}{B^1(\Gamma; M)}.$$

We need to specify the action Shimura uses on the vector space W^{2k}. If $k = 0$, then $SL_2(\mathbb{Z})$ acts trivially on $W^0 = \mathbb{R}$. For $k > 0$, let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and define an action on elements of the form

$$\sum_{r=0}^{k} u^{k-r} v^r x^{k-r} y^r$$

where $u, v \in \mathbb{R}$ by

$$\alpha \cdot \left(\sum_{r=0}^{k} u^{k-r} v^r x^{k-r} y^r \right) = \sum_{r=0}^{k} (au + bv)^{k-r} (cu + dv)^r x^{k-r} y^r.$$

We will show that this action can be extended to all of W^{2k} by proving that the action is isomorphic to the polynomial replacement action.

Let $V = \mathbb{R}[x, y]$ with the polynomial replacement action. We can define a $\mathbb{R}$-vector space isomorphism $f: W^{2k} \to V^{2k}$ by

$$f\left(\sum_{r=0}^{k} c_r x^{k-r} y^r\right) = \sum_{r=0}^{k} \binom{k}{r} c_r x^{k-r} y^r.$$

Proposition 9. *The map $f: W^{2k} \to V^{2k}$ respects the action of $SL_2(\mathbb{Z})$ on W and V. That is, if $\alpha \in SL_2(\mathbb{Z})$ then*

$$f\left(\alpha \cdot \sum_{r=0}^{k} u^{k-r} v^r x^{k-r} y^r\right) = \alpha \cdot f\left(\sum_{r=0}^{k} u^{k-r} v^r x^{k-r} y^r\right).$$

Proof. Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The key point is that

$$f\left(\alpha \cdot \sum_{r=0}^{k} u^{k-r} v^r x^{k-r} y^r\right) = (aux + bvx + cuy + dvy)^k$$

$$= \alpha \cdot f\left(\sum_{r=0}^{k} u^{k-r} v^r x^{k-r} y^r\right). \qquad \square$$

Corollary 10. *Shimura's action extends to all of $\mathbb{R}^{m+1}$. That is, if $w \in W$, then we can define*

$$\alpha \cdot w = f^{-1}(\alpha \cdot f(w)).$$

Thus, the map $f: W^{2m} \to V^{2m}$ is a $SL_2(\mathbb{Z})$-module isomorphism.

We can now define the $\mathbb{R}$-vector space map $\varphi: M_{k+2}^{\mathbb{C}}(\Gamma) \to H^1(\Gamma; W^{2k})$. If $f \in M_{k+2}^{\mathbb{C}}(\Gamma)$, let $\varphi(f)$ be the class of the derivation $\omega_f : \Gamma \to W^{2k}$ defined as follows: Fix $z_0 \in \mathscr{H}$. Given $\gamma \in \Gamma$, define $c_r \in \mathbb{R}$ by the complex line integral

$$c_r = \int_{z_0}^{\gamma \cdot z_0} \operatorname{Re}(f(z) z^{k-r} \mathrm{d}z). \tag{4}$$

Then define $\omega_f(\gamma) \in W^{2k}$ by

$$\omega_f(\gamma) = \sum_{r=0}^{k} c_r x^{k-r} y^r. \tag{5}$$

Shimura shows that ω_f is a derivation [10, p. 233], and thus defines an element of $H^1(\Gamma; W^{2k})$.

However, this map is not an isomorphism. Shimura restricts the map to $S_{k+2}^{\mathbb{C}}(\Gamma)$ and shows that φ maps into the parabolic cohomology and defines $\mathbb{R}$-vector space isomorphism between the cusp forms and the parabolic cohomology:

$$\varphi : S^{\mathbb{C}}_{k+2}(\Gamma) \xrightarrow{\cong} H^1_p(\Gamma; W^{2k}).$$

4. The isomorphism between X_Γ and modular forms

We are now ready to show that for certain congruence subgroups, X_Γ has a stable summand X_{Γ^-} where

$$\varphi : M^{\mathbb{R}}_{2k+2}(\Gamma) \to H^{4k+1}(X_{\Gamma^-}; \mathbb{R})$$

is an isomorphism for $k > 0$.

Until otherwise noted, we will assume that $k > 0$. Shimura calculates the dimensions of $S^{\mathbb{C}}_{2k+2}(\Gamma)$ and $M^{\mathbb{C}}_{2k+2}(\Gamma)$ as $\mathbb{C}$-vector spaces [10, p. 46]. In particular, let $s_1, s_2, \ldots, s_m$ be representatives of the cusps of Γ. Then he shows that

$$dim_{\mathbb{C}}\, M^{\mathbb{C}}_{2k+2}(\Gamma) = dim_{\mathbb{C}}\, S^{\mathbb{C}}_{2k+2}(\Gamma) + m.$$

Therefore, we have a short exact sequence

$$0 \to S^{\mathbb{C}}_{2k+2}(\Gamma) \to M^{\mathbb{C}}_{2k+2}(\Gamma) \to \oplus^m_1 \mathbb{C} \to 0. \tag{6}$$

We have an analogous exact sequence involving parabolic cohomology. Let $\gamma_i \in \Gamma$ be a parabolic element that generates Γ_{s_i}, the stabilizer of the cusp s_i. From [4], we have a short exact sequence for $k > 0$,

$$0 \to H^1_p(\Gamma; V^{4k}) \to H^1(\Gamma; V^{4k}) \to \oplus^m_{i=1} H^1(\langle\gamma_i\rangle; V^{4k}) \to 0. \tag{7}$$

Proposition 11. *If $\gamma \in \Gamma$ is parabolic, then $H^1(\langle\gamma\rangle; V^{4k}) \cong \mathbb{R}$.*

Proof. We will consider any class in $H^1(\langle\gamma\rangle; V^{4k})$ to be represented by a derivation $\langle\gamma\rangle \to V^{4k}$. Since $\langle\gamma\rangle$ is infinite cyclic, any derivation is completely determined by its value on γ. Thus,

$$H^1(\langle\gamma\rangle; V^{4k}) \cong \frac{Der(\langle\gamma\rangle, V^{4k})}{InDer(\langle\gamma\rangle, V^{4k})} \cong \frac{V^{4k}}{(\gamma - 1)\cdot V^{4k}}$$

and

$$\begin{aligned} dim_{\mathbb{R}}\, H^1(\langle\gamma\rangle; V^{4k}) &= dim_{\mathbb{R}}\, V^{4k} - dim_{\mathbb{R}}\, (\gamma - 1)\cdot V^{4k} \\ &= dim_{\mathbb{R}}\, K, \end{aligned}$$

where $K = ker\{V^{4k} \xrightarrow{\gamma - 1} (\gamma - 1)\cdot V^{4k}\} = (V^{4k})^\gamma$.

Let s be the cusp stabilized by γ, and let $\rho \in SL_2(\mathbb{Z})$ be such that $\rho \cdot s = \infty$. Then $\rho\gamma\rho^{-1}$ fixes ∞. Thus $\rho\gamma\rho^{-1} = \gamma^h_\infty$ for some h, and $(V^{4k})^\gamma$ is isomorphic to $(V^{4k})\gamma^h_\infty$. But we know that $(V^{4k})\gamma^h_\infty \cong \mathbb{R}\{x^{2k}\}$ by Lemma 1, so $1 = dim_{\mathbb{R}}\ K = dim_{\mathbb{R}}$ $H^1(\langle\gamma\rangle; V^{4k})$, and $H^1(\langle\gamma\rangle; V^{4k}) \cong \mathbb{R}$. $\square$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
S^{\mathbb{C}}_{2k+2}(\Gamma) \arrow[r, "\cong"] \arrow[d] & H^1_p(\Gamma; V^{4k}) \arrow[d] \\
M^{\mathbb{C}}_{2k+2}(\Gamma) \arrow[r, "\varphi"] \arrow[d] & H^1(\Gamma; V^{4k}) \arrow[d] \\
\oplus^m_1 \mathbb{C} \arrow[r] \arrow[d] & \oplus^m_1 \mathbb{R} \arrow[d] \\
0 & 0
\end{tikzcd}$$

Fig. 1. The initial diagram for $k > 0$.

Combining the short exact sequences (6), (7), and the Eichler–Shimura map, we have the commutative diagram in Fig. 1 for $k > 0$. Nishida proposes how to amend the map φ to form an isomorphism [8].

Let $\sigma = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, and let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup such that σ acts on Γ by conjugation, such as $\Gamma(N)$, $\Gamma_0(N)$, or $SL_2(\mathbb{Z})$. We can let σ act on $(\mathbb{C}P^\infty)^{\times 2}$ as usual, but for σ to act on $X_\Gamma = E\Gamma \times_\Gamma (\mathbb{C}P^\infty)^{\times 2}$, we must check that it respects the Γ action on $(\mathbb{C}P^\infty)^{\times 2}$. In fact, this holds for arbitrary $\tau \in GL_2(\mathbb{Z})$ that act on Γ by conjugation.

Proposition 12. *If $\tau \in GL_2(\mathbb{Z})$ acts on Γ by conjugation, then τ acts on X_Γ.*

Proof. Let $w = (g, z) \in X_\Gamma$. We need to show that if $\gamma \in \Gamma$, then $(w \cdot \gamma) \cdot \tau = (w \cdot \tau) \cdot \gamma'$ for some $\gamma' \in \Gamma$. Then

$$\begin{aligned}
(w \cdot \gamma) \cdot \tau &= (g\gamma, z \cdot \gamma) \cdot \tau \\
&= (\tau^{-1} g \gamma \tau, z \cdot \gamma \tau) \\
&= (\tau^{-1} g \gamma \tau \gamma', z \cdot \tau \gamma') \quad \text{where } \tau^{-1}\gamma\tau = \gamma' \in \Gamma \\
&= (w \cdot \tau) \cdot \gamma'. \qquad \square
\end{aligned}$$

Corollary 13. *Consider X_Γ as a suspension spectrum, and suppose that we have inverted 2. Then*

$$X_\Gamma \cong X_{\Gamma^-} \vee X_{\Gamma^+}$$

where X_{Γ^-} and X_{Γ^+} are the -1 and $+1$ eigenspaces of σ, respectively.

Proof. Since σ has order two,

$$e_0 = \frac{1-\sigma}{2} \quad \text{and} \quad e_1 = \frac{1+\sigma}{2}$$

define primitive orthogonal idempotents in $\mathbb{Z}(\frac{1}{2})[\sigma]$. Since we have inverted 2 and σ acts on X_Γ, these idempotents also act on X_Γ. Thus,

$$X_\Gamma \cong e_0 \cdot X_\Gamma \vee e_1 \cdot X_\Gamma.$$

Let $w \in X_\Gamma$, and notice that $e_0 \cdot w = w$ if and only if $\sigma \cdot w = -w$. Thus, $e_0 \cdot X_\Gamma = X_{\Gamma^-}$. In the same manner, $e_1 \cdot X_\Gamma = X_{\Gamma^+}$. □

Notice that $H^{4k+1}(X_{\Gamma^-}; R) \cong (H^1(\Gamma; U^{4k}))^-$. We want to show that the short exact sequence (7) respects the action of σ. First, we need to show that σ acts on each module in the sequence.

Proposition 14. *Suppose σ acts on Γ by conjugation, and let M be a Γ-module on which σ also acts. Then σ acts on $H^1_p(\Gamma; M)$.*

Proof. Let f be a parabolic cocycle. We will show that $\sigma \cdot f$ is also in $Z^1_p(\Gamma; M)$. Let $\gamma \in \Gamma$ be a parabolic element fixing $s \in \mathbb{Q}$. Then $\sigma\gamma\sigma$ is a parabolic element of Γ fixing $\sigma \cdot s$. Since f is a parabolic cocycle, $f(\sigma\gamma\sigma) = (\sigma\gamma\sigma - 1)\cdot m$ for some $m \in M$. Then

$$\begin{aligned}(\sigma \cdot f)(\gamma) &= \sigma \cdot f(\sigma\gamma\sigma) = \sigma(\sigma\gamma\sigma - 1)\cdot m \\ &= (\gamma - 1)\sigma \cdot m \in (\gamma - 1)\cdot M.\end{aligned}$$

Thus, $\sigma \cdot f \in Z^1_p(\Gamma; M)$ and σ acts on $H^1_p(\Gamma; M)$. □

Therefore, σ acts on $H^1_p(\Gamma; U^{4k})$, and we let $\bigoplus_{i=1}^m H^1(\langle\gamma_i\rangle; V^{4k})$ inherit the action of σ so that the short exact sequence (7) is a sequence of σ modules.

Corollary 15. *We have a short exact sequence*

$$0 \to H^1_p(\Gamma; V^{4k})^- \to H^1(\Gamma; V^{4k})^- \to (\textstyle\bigoplus_{i=1}^m \mathbb{R})^- \to 0.$$

Proof. Since σ has order 2, $\mathbb{Z}[\frac{1}{2}][\sigma]$ is a semi-simple ring with primitive orthogonal idempotents e_0, e_1. Therefore, any module M decomposes naturally as

$$M = e_0 \cdot M \oplus e_1 \cdot M = M^- \oplus M^+. \qquad \square$$

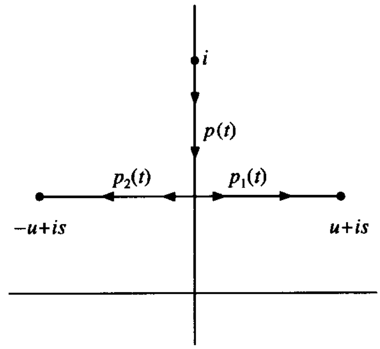

Fig. 2. Paths of line integrals.

Proposition 16. *Let φ denote the Eichler–Shimura map. If $f \in M^{\mathbb{R}}_{2k+2}(\Gamma)$, then*

$$\varphi(f) \in H^1(\Gamma; V^{4k})^-,$$

and if $f \in S^{\mathbb{R}}_{2k+2}(\Gamma)$, then

$$\varphi(f) \in H^1_p(\Gamma; V^{4k})^-.$$

Proof. If we show the claim for $M^{\mathbb{R}}_{2k+2}(\Gamma)$, then the claim for $S^{\mathbb{R}}_{2k+2}(\Gamma)$ will follow since $S^{\mathbb{R}}_{2k+2}(\Gamma) \subseteq M^{\mathbb{R}}_{2k+2}(\Gamma)$ and φ maps the cusp forms into the parabolic cohomology. Recall the definition of $\varphi(f)$ given in (4) and (5): $\varphi(f) = [\omega_f]$ where $\omega_f : \Gamma \to V^{4k}$ is a derivation defined by

$$\omega_f(\gamma) = \sum_{r=0}^{2k} c_r x^{2k-r} y^r \quad \text{where } c_r = \int_{z_0}^{\gamma(z_0)} \operatorname{Re}(f(z) z^{k-r}\, \mathrm{d}z).$$

We want to show that $\sigma \cdot \omega_f$ and $-\omega_f$ differ by at most a coboundary. In fact, we will show that they are equal.

Recall that $(\sigma \cdot \omega_f)(\gamma) = \sigma \cdot \omega_f(\sigma\gamma\sigma)$, since σ acts on Γ by conjugation. Let $\omega_f(\sigma\gamma\sigma) = \sum_{r=0}^{2k} b_r x^{2k-r} y^r$ where b_r is the appropriate path of the line integral for $\sigma\gamma\sigma$. Then

$$\begin{aligned}(\sigma \cdot \omega_f)(\gamma) &= \sigma \cdot \omega_f(\sigma\gamma\sigma) = \sigma \cdot \left(\sum_{r=0}^{2k} b_r x^{2k-r} y^r \right) \\ &= \sum_{r=0}^{2k} b_r x^{2k-r} (-y)^r = \sum_{r=0}^{2k} (-1)^r b_r x^{2k-r} y^r.\end{aligned}$$

Thus, we want to show that $(-1)^r b_r = -c_r$. Then, if r is odd, we want $b_r = c_r$, and if r is even, we want $b_r = -c_r$.

Since z_0 is arbitrary, we can choose $z_0 = i$ to help calculate the line integrals. Notice that for any γ, $\sigma\gamma\sigma \cdot i = -\overline{\gamma \cdot i}$. Let $\gamma \cdot z_0 = u + is$, so $\sigma\gamma\sigma \cdot z_0 = -u + is$, and the paths look something like Fig. 2.

Since f is a real modular form, we know that the q-expansion of f has real coefficients. Let $f(z) = \sum_{j=0}^{\infty} a_j q_h^j$ where $a_j \in \mathbb{R}$ and $q_h = e^{2\pi i z/h}$. We will consider the vertical and horizontal parts of the integrals separately, and show that we have the desired conditions.

We can parametrize the vertical part of the paths by

$$p(t) = (1-t)\mathrm{i} + t\mathrm{i}s, \quad 0 \le t \le 1.$$

Then the integrals become

$$\int_{i}^{is} \mathrm{Re}(f(z)z^{2k-r}\,\mathrm{d}z) = \int_0^1 \mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(2\pi\mathrm{i}j/h)(p(t))} p(t)^{2k-r} p'(t)\,\mathrm{d}t\right)$$

$$= \int_0^1 \mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(-2\pi j/h)((1-t)+ts)} \mathrm{i}^{2k-r+1}((1-t)+ts)^{2k-r}(-1+s)\,\mathrm{d}t\right)$$

The integrand is purely real except for the i^{2k-r+1} term. Thus, if r is odd, we have that b_r and c_r agree on $p(t)$. If r is even, the integrand is 0, giving $b_r = -c_r$ on $p(t)$. So we have $(-1)^r \int_p = -\int_p$.

Now, consider the horizontal parts of the paths. For c_r, we can parametrize this part of the path by

$$p_1(t) = tu + \mathrm{i}s, \quad 0 \le t \le 1.$$

For b_r, we can parametrize the horizontal part by

$$p_2(t) = -tu + \mathrm{i}s, \quad 0 \le t \le 1.$$

For c_r, the integrand becomes

$$\mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(2\pi\mathrm{i}j/h)(tu+\mathrm{i}s)}(tu+\mathrm{i}s)^{2k-r} u\,\mathrm{d}t\right)$$

$$= \mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(-2\pi js/h)} \mathrm{e}^{(2\pi\mathrm{i}jtu/h)}(tu+\mathrm{i}s)^{2k-r} u\,\mathrm{d}t\right)$$

$$= \mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(-2\pi js/h)} z_1 z_2^{2k-r} u\,\mathrm{d}t\right),$$

where $z_1 = \mathrm{e}^{(-2\pi js/h)}$ and $z_2 = tu + \mathrm{i}s$. For b_r, we get

$$\mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(2\pi\mathrm{i}j/h)(-tu+\mathrm{i}s)}(-tu+\mathrm{i}s)^{2k-r}(-u)\,\mathrm{d}t\right)$$

$$= -\mathrm{Re}\left(\sum_{j=0}^{\infty} a_j \mathrm{e}^{(-2\pi js/h)} \mathrm{e}^{(-2\pi\mathrm{i}jtu/h)}(-tu+\mathrm{i}s)^{2k-r} u\,\mathrm{d}t\right)$$

$$= -\mathrm{Re}\left(\sum_{j=0}^{\infty} a_j e^{(-2\pi j s/h)} \overline{z_1}(-\overline{z_2})^{2k-r} u\,dt\right)$$

$$= -\mathrm{Re}\left(\sum_{j=0}^{\infty} a_j e^{(-2\pi j s/h)} (-1)^{2k-r} \overline{z_1 z_2^{2k-r}}\, u\,dt\right)$$

$$= (-1)^{2k-r+1}\,\mathrm{Re}\left(\sum_{j=0}^{\infty} a_j e^{(-2\pi j s/h)} z_1 z_2^{2k-r} u\,dt\right).$$

The last equality holds since $\mathrm{Re}(z) = \mathrm{Re}(\bar{z})$. Therefore, the integrands differ by a factor of $(-1)^{2k-r+1}$ and $(-1)^r \int_{p_2} = -\int_{p_1}$. Thus, $(-1)^r b_r = -c_r$. □

Thus, we have an algebraic map

$$M_{2k+2}^{\mathbb{R}}(\Gamma) \to H^1(\Gamma; V^{4k})^- \cong H^{4k+1}(X_{\Gamma^-}; \mathbb{R}).$$

We will show that this is an isomorphism if the complex forms of Γ are generated as a $\mathbb{R}$-vector space by the real forms. However, this condition is not overly restrictive. Shimura shows for $\Gamma = \Gamma_0(N)$ or $\Gamma = SL_2(\mathbb{Z})$ that the $\mathbb{Q}$ cusp forms generate the $\mathbb{C}$ cusp forms as a $\mathbb{C}$-vector space [10, Theorem 3.52, p. 85], i.e.

$$S_*^{\mathbb{Q}}(\Gamma) \otimes_{\mathbb{Q}} \mathbb{C} \cong S_*^{\mathbb{C}}(\Gamma).$$

Further, the non-cusp forms of $\Gamma_0(N)$ and $SL_2(\mathbb{Z})$ are generated by the Eisenstein series [10, p. 78]. Since the q-expansion of the Eisenstein series have rational coefficients, we have

$$M_*^{\mathbb{Q}}(\Gamma) \otimes_{\mathbb{Q}} \mathbb{C} \cong M_*^{\mathbb{C}}(\Gamma)$$

for $\Gamma = \Gamma_0(N)$ or $\Gamma = SL_2(\mathbb{Z})$.

Proposition 17. *Suppose that the complex cusp forms of Γ are generated as a $\mathbb{R}$-vector space by the real forms. Then the Eichler–Shimura map defines an isomorphism*

$$S_{2k+2}^{\mathbb{R}}(\Gamma) \xrightarrow{\cong} H_p^1(\Gamma; V^{4k})^-.$$

Proof. Since $S_{2k+2}^{\mathbb{C}}(\Gamma)$ is generated as a $\mathbb{C}$-vector space by $S_{2k+2}^{\mathbb{R}}(\Gamma)$, we have

$$S_{2k+2}^{\mathbb{C}}(\Gamma) \cong S_{2k+2}^{\mathbb{R}}(\Gamma) \oplus \mathrm{i}S_{2k+2}^{\mathbb{R}}(\Gamma)$$

as $\mathbb{R}$-vector spaces. Since $S_{2k+2}^{\mathbb{C}}(\Gamma) \cong H_p^1(\Gamma; V^{4k})$, it will suffice to show that $\mathrm{i}S_{2k+2}^{\mathbb{R}}(\Gamma)$ maps into $H_p^1(\Gamma; V^{4k})^+$. In fact, we will show that $\mathrm{i}M_{2k+2}^{\mathbb{R}}(\Gamma)$ maps into $H^1(\Gamma; V^{4k})^+$.

Let $f \in M_{2k+2}^{\mathbb{R}}(\Gamma)$, so we want to show that $\sigma \cdot \omega_{\mathrm{i}f}$ and $\omega_{\mathrm{i}f}$ differ by at most a coboundary. As before, we will show equality. Let b_r be the coefficients for $\sigma \cdot \omega_{\mathrm{i}f}$ and c_r be the coefficients for $\omega_{\mathrm{i}f}$. Following the same analysis as in the proof of Proposition 16, we want $(-1)^r b_r = c_r$. If r is odd, we want $b_r = -c_r$; and if r is even, we want $b_r = c_r$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
S^{\mathbb{R}}_{2k+2}(\Gamma) \arrow[r, "\cong"] \arrow[d] & H^1_p(\Gamma; V^{4k})^- \arrow[d] \\
M^{\mathbb{R}}_{2k+2}(\Gamma) \arrow[r, "\varphi"] \arrow[d] & H^1(\Gamma; V^{4k})^- \arrow[d] \\
{}^m_1\mathbb{R} \arrow[r, "\bar{\varphi}"] \arrow[d] & (\oplus^m_1 \mathbb{R})^- \arrow[d] \\
0 & 0
\end{tikzcd}$$

Fig. 3. The amended diagram for $k > 0$.

Notice that the only change in the integrands is that the q-expansion of if has purely imaginary coefficients:

$$\mathrm{i}f(z) = \sum_{j=0}^{\infty} \mathrm{i}a_j q_h^j.$$

On $p(t)$ we get the same integrand, except for the factor i^{2k-r+2} instead of i^{2k-r+1}. Then, if r is odd, the integrand is 0, so that $b_r = -c_r$. If r is even, we get $b_r = c_r$.

On $p_1(t)$ the integrand for c_r becomes

$$\mathrm{Re}\left(\sum_{j=0}^{\infty} \mathrm{i}a_j \mathrm{e}^{(-2\pi js/h)} z_1 z_2^{2k-r} u \, \mathrm{d}t \right).$$

And on $p_2(t)$ the integrand for b_r becomes

$$(-1)^{2k-r+1}\, \mathrm{Re}\left(\sum_{j=0}^{\infty} \mathrm{i}a_j \mathrm{e}^{(-2\pi js/h)} \overline{z_1 z_2^{2k-r}} u \, \mathrm{d}t \right).$$

Since $\mathrm{Re}(\mathrm{i}\bar{z}) = -\mathrm{Re}(\mathrm{i}z)$, the integrands differ by a factor of $(-1)^{2k-r+2}$. Thus, $b_r = (-1)^r c_r$. $\square$

This gives us the diagram in Fig. 3. If we show that the bottom map $\bar{\varphi}$ is injective, then it must be an isomorphism for dimensional reasons, and φ will be an isomorphism by the Five Lemma.

Proposition 18. *Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup on which σ acts by conjugation, and suppose that the complex forms of Γ are generated as a $\mathbb{R}$-vector space by the real forms. If Γ has a single cusp at ∞ or two cusps at 0 and ∞, then $\bar{\varphi}$ is injective.*

Proof. Let $f \in M^{\mathbb{R}}_{2k+2}(\Gamma)$ be a non-cusp form. We will show $\varphi(f) \notin H^1_p(\Gamma; V^{4k})^-$, giving $\bar{\varphi}$ is injective. Since f is not a cusp form, then either the q-expansion or the expansion at 0 has a non-zero constant term. We will treat each case separately.

Suppose that the q-extension of f has a non-zero constant term. Then from (2),

$$f(z) = \sum_{j=0}^{\infty} a_j q_h^j \quad \text{where } q_h = e^{(2\pi i z/h)} \quad \text{and } a_0 \neq 0.$$

If $\gamma \in \Gamma$ is the generator of Γ_∞, then γ is parabolic and $\gamma = \gamma_\infty^h$. We will show that $\omega_f(\gamma) \notin (\gamma - 1) \cdot V^{4k}$.

First, we show that no element of $(\gamma - 1) \cdot V^{4k}$ contains a y^{2k} term. Recall that $\gamma \cdot x^{2k-r} y^r = x^{2k-r}(hx + y)^r$. Then

$$\begin{aligned}(\gamma - 1) \cdot x^{2k-r} y^r &= x^{2k-r}(hx + y)^r - x^{2k-r} y^r \\ &= (x^{2k-r} y^r + \text{lower powers of } y) - x^{2k-r} y^r.\end{aligned}$$

Thus, $\gamma - 1$ always lowers the power of y, and $(\gamma - 1) \cdot V^{4k}$ has no element containing y^{2k}.

Therefore, it suffices to show that the coefficient c_{2k} of the y^{2k}-term in $\omega_f(\gamma)$ is non-zero. Then,

$$c_{2k} = \int_{z_0}^{\gamma \cdot z_0} \operatorname{Re}(f(z)\,dz) = \int_{z_0}^{\gamma \cdot z_0} \operatorname{Re}\left(\sum_{j=0}^{\infty} a_j q_h^j \,dz \right).$$

As before, let $z_0 = i$. Then $\gamma \cdot z_0 = i + h$ and we can parametrize the path by $p(t) = i + th$, $0 \le t \le 1$. Therefore, the integral becomes

$$\begin{aligned} c_{2k} &= \int_0^1 \operatorname{Re}\left(\sum_{j=0}^{\infty} a_j e^{(2\pi i j/h)(i + th)} h \,dt \right) \\ &= \int_0^1 \operatorname{Re}\left(\sum_{j=0}^{\infty} a_j e^{(-2\pi j/h)} e^{2\pi i j t} h \,dt \right) \\ &= \int_0^1 a_0 h \,dt + \sum_{j=1}^{\infty} a_j e^{(-2\pi j/h)} h \operatorname{Re}\left(\int_0^1 e^{2\pi i j t} \,dt \right). \end{aligned}$$

However, $\int_0^1 e^{2\pi i j t}\,dt = 0$, so we have

$$c_{2k} = \int_0^1 a_0 h \,dt = a_0 h \neq 0$$

since $a_0 \neq 0$ by hypothesis. Therfore, if the q-expansion of f has non-zero constant term, then ω_f is not a parabolic cocyle.

Suppose that the expansion of f at 0 has a non-zero constant term. As in (3) let $\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that

$$f \,|\, [\rho]_{2k+2}(z) = \sum_{j=0}^{\infty} b_j q_h^j \quad \text{where } q_h = \mathrm{e}^{(2\pi \mathrm{i} z/h)} \text{ and } b_0 \neq 0.$$

If $\gamma \in \Gamma$ is the generator of Γ_0, then γ is parabolic and $\gamma = \gamma_0^h$. As before, we will show that $\omega_f(\gamma) \notin (\gamma - 1) \cdot V^{4k}$.

A similar analysis of $(\gamma - 1) \cdot x^{2k-r} y^r$ as a above shows that no element of $(\gamma - 1) \cdot V^{4k}$ has a term involving x^{2k}. Thus, it suffices to show that the coefficient of the x^{2k}-term in $\omega_f(\gamma)$ is non-zero. The coefficient of x^{2k} is

$$c_0 = \int_{z_0}^{\gamma \cdot z_0} \mathrm{Re}(f(z) z^{2k}\, dz).$$

First, consider the change of variable $z = \rho^{-1} \cdot \tau = (-1/\tau)$ and $\mathrm{d}z = (1/\tau^2)\,\mathrm{d}\tau$. Then we have

$$\begin{aligned} c_0 &= \int_{\rho \cdot z_0}^{\rho\gamma \cdot z_0} \mathrm{Re}\left(f\left(-\frac{1}{\tau}\right)\left(-\frac{1}{\tau}\right)^{2k} \frac{1}{\tau^2}\,\mathrm{d}\tau \right) \\ &= \int_{\rho \cdot z_0}^{\rho\gamma \cdot z_0} \mathrm{Re}\left(f\left(-\frac{1}{\tau}\right) \tau^{-(2k+2)}\,\mathrm{d}\tau \right) \\ &= \int_{\rho \cdot z_0}^{\rho\gamma \cdot z_0} \mathrm{Re}(f \,|\, [\rho]_{2k+2}(\tau)\,\mathrm{d}\tau) \\ &= \int_{\rho \cdot z_0}^{\rho\gamma \cdot z_0} \mathrm{Re}\left(\sum_{j=0}^{\infty} b_j q_h^j\,\mathrm{d}\tau \right). \end{aligned}$$

As above, let $z_0 = \mathrm{i}$ so that $\rho \cdot z_0 = (-1/\mathrm{i}) = \mathrm{i}$, $\rho\gamma \cdot z_0 = \rho \cdot (\mathrm{i}/(\mathrm{i}h + 1)) = -(\mathrm{i}h + 1)/\mathrm{i} = \mathrm{i} - h$. Then we can parametrize the path by $p(t) = \mathrm{i} - th$, $0 \le t \le 1$. Therefore, the integral becomes

$$\begin{aligned} c_0 &= \int_0^1 \mathrm{Re}\left(\sum_{j=0}^{\infty} b_j \mathrm{e}^{(2\pi \mathrm{i} j(\mathrm{i} - th)/h)}(-h)\,\mathrm{d}t \right) \\ &= \int_0^1 \mathrm{Re}\left(\sum_{j=0}^{\infty} b_j \mathrm{e}^{-(2\pi j/h)} \mathrm{e}^{-(2\pi \mathrm{i} jth/h)}(-h)\,\mathrm{d}t \right) \\ &= \int_0^1 \mathrm{Re}(b_0(-h)\,\mathrm{d}t + \sum_{j=1}^{\infty} \mathrm{e}^{-(2\pi j/h)}\, \mathrm{Re}\left(b_j \int_0^1 \mathrm{e}^{-2\pi \mathrm{i} jt}\,\mathrm{d}t \right). \end{aligned}$$

As above, $\int_0^1 \mathrm{e}^{-2\pi \mathrm{i} jt}\,\mathrm{d}t = 0$, so we have

$$c_0 = \int_0^1 \mathrm{Re}(b_0(-h)\,\mathrm{d}t) = -h\,\mathrm{Re}(b_0).$$

But $\mathrm{Re}(b_0) \neq 0$ since

$$\begin{aligned} b_0 &= \lim_{q_h \to 0} \sum_{j=0}^{\infty} b_j q_h^j \\ &= \lim_{y \to \infty} \sum_{j=0}^{\infty} b_j \mathrm{e}^{(2\pi \mathrm{i}(\mathrm{i}y)/h)} \\ &= \lim_{y \to \infty} f | [\rho]_{2k+2}(\mathrm{i}y) \\ &= \lim_{y \to \infty} f\left(-\frac{1}{\mathrm{i}y}\right)(\mathrm{i}y)^{-(2k+2)} \\ &= \lim_{y \to \infty} \sum_{j=0}^{\infty} a_j \mathrm{e}^{2\pi \mathrm{i} j(-1/\mathrm{i}y)} (\mathrm{i}y)^{-(2k+2)} \\ &= \lim_{y \to \infty} \sum_{j=0}^{\infty} a_j \mathrm{e}^{2\pi j(-1/y)} (-1)^{k+1} y^{-(2k+2)} \in \mathbb{R}. \end{aligned}$$

That is, f is a real modular form, so its q-expansion has real coefficients. Since $b_0 \neq 0$ by hypothesis, $b_0 \in \mathbb{R}$ implies $\mathrm{Re}(b_0) \neq 0$. Therefore, $c_0 = -h\,\mathrm{Re}(b_0) \neq 0$ giving that ω_f is not a parabolic cocycle. $\square$

Theorem 19. *Let Γ be a congruence subgroup on which σ acts by conjugation. Suppose*

(1) *$M_{2k}^{\mathbb{C}}(\Gamma)$ is generated as a $\mathbb{R}$-vector space by $M_{2k}^{\mathbb{R}}(\Gamma)$.*

(2) *Γ has a single cusp at ∞, or two cusps at ∞ and 0.*

Then the Eichler–Shimura map restricts to an isomorphism

$$\varphi : M_{2k+2}^{\mathbb{R}}(\Gamma) \to H^1(\Gamma; V^{4k})^-$$

for $k > 0$, and thus

$$M_{2k+2}^{\mathbb{R}}(\Gamma) \cong H^{4k+1}(X_{\Gamma^-}; \mathbb{R}).$$

In particular, $SL_2(\mathbb{Z})$ and $\Gamma_0(p)$ satisfy the hypotheses of the theorem. Shimura [10, Ch. 2] expresses $dim_{\mathbb{C}}\, S_{2k+2}^{\mathbb{C}}(\Gamma_0(p))$ in terms of p and k. Thus, Theorem 19 and Fig. 3 make it possible to express $dim_{\mathbb{R}}\, H^{4k+1}(X_{\Gamma_0(p)}^-; \mathbb{R})$ purely in terms of p and k. Shimura's calculations also permit us to perform similar calculations for $X_{SL_2(\mathbb{Z})}^-$ that depend solely on k.

We will now show that Theorem 19 holds for $SL_2(\mathbb{Z})$ and $\Gamma_0(2)$ when $k = 0$. Notice that Proposition 16 holds for $k = 0$, but we do not necessarily have the short exact sequence (7).

Proposition 20. *$H^1(SL_2(\mathbb{Z}); \mathbb{R}) = 0$ and $M_2^{\mathbb{R}}(SL_2(\mathbb{Z})) = 0$. Therefore,*

$$\varphi : M_{2k+2}^{\mathbb{R}}(SL_2(\mathbb{Z})) \xrightarrow{\cong} H^{4k+1}(X_{SL_2(\mathbb{Z})}^-; \mathbb{R})$$

for all $k \geq 0$.

Proof. Recall that $SL_2(\mathbb{Z}) \cong \langle\gamma_4\rangle *_{\langle -Id\rangle} \langle\gamma_6\rangle$ where $\gamma_4 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\gamma_6 = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Therefore, we have a Mayer–Vietoris sequence in cohomology

$$\begin{aligned} 0 &\to H^0(SL_2(\mathbb{Z}); \mathbb{R}) \to H^0(\langle\gamma_4\rangle; \mathbb{R}) \oplus H^0(\langle\gamma_6\rangle; \mathbb{R}) \\ &\to H^0(\langle -Id\rangle; \mathbb{R}) \to H^1(SL_2(\mathbb{Z}); \mathbb{R}) \\ &\to H^1(\langle\gamma_4\rangle; \mathbb{R}) \oplus H^1(\langle\gamma_6\rangle; \mathbb{R}). \end{aligned}$$

Since $SL_2(\mathbb{Z})$ acts trivially on $\mathbb{R}$, all of the zero-dimensional cohomology groups are isomorphic to $\mathbb{R}$, and since $\langle\gamma_4\rangle$ and $\langle\gamma_6\rangle$ are finite cyclic,

$$H^1(\langle\gamma_4\rangle; \mathbb{R}) \oplus H^1(\langle\gamma_6\rangle; \mathbb{R}) = 0 \oplus 0 = 0.$$

Thus, we have a sequence of $\mathbb{R}$-vector spaces

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to H^1(SL_2(\mathbb{Z}); \mathbb{R}) \to 0.$$

Therefore, $H^1(SL_2(\mathbb{Z}); \mathbb{R}) = 0$.

Furthermore, Shimura [10, Proposition 2.26, p. 48] shows that $M_2^{\mathbb{C}}(SL_2(\mathbb{Z})) = 0$. Therefore, $M_2^{\mathbb{R}}(SL_2(\mathbb{Z})) = 0$ and the rest of the proposition follows. □

We should also point out that we have a Mayer–Vietoris sequence with coefficients in U^{4k} that reduces to

$$0 \to (U^{4k})^{\gamma_4} \oplus (U^{4k})^{\gamma_6} \to U^{4k} \to H^1(SL_2(\mathbb{Z}); U^{4k}) \to 0.$$

From this sequence, it is possible to construct explicit generators in $H^1(SL_2(\mathbb{Z}); U^{4k})$.

We will now show that Theorem 19 holds for $\Gamma_0(2)$ and $k = 0$. Let $R' = \mathbb{Z}(\frac{1}{2})$. First, we will show that $H^1(\Gamma_0(2); R') = R'$.

Lemma 21.

$$H^1(\Gamma_0(2); R') \cong H^1(\overline{\Gamma_0(2)}; R').$$

Proof. We have a short exact sequence

$$1 \to \langle -Id\rangle \to \Gamma_0(2) \to \overline{\Gamma_0(2)} \to 1$$

and an assoicated Serre spectral sequence with

$$E_2^{r,s} = H^r(\overline{\Gamma_0(2)}; H^s(\langle -Id\rangle; R')).$$

Since $\langle -Id \rangle \cong \mathbb{Z}/2$ and $\frac{1}{2} \in R'$, $H^s(\langle -Id \rangle; R') = 0$ for all $s > 0$, and $\langle -Id \rangle$ acts trivially on R', giving $H^0(\langle -Id \rangle; R') \cong R'$. Thus, the spectral sequence reduces to

$$E_2^{r,0} = H^r(\overline{\Gamma_0(2)}: R').$$

There are no differentials and

$$H^1(\Gamma_0(2); R') \cong H^1(\overline{\Gamma_0(2)}; R'). \qquad \square$$

Since $[SL_2(\mathbb{Z}); \Gamma(2)] = 6$ and $[SL_2(\mathbb{Z}); \Gamma_0(2)] = 3$ [10, Proposition 1.43, p. 24], we have $[\Gamma_0(2); \Gamma(2)] = 2$. We will choose γ_∞ as the generator of $\Gamma_0(2)/\Gamma(2) \cong \mathbb{Z}/2$. We will compute $H^1(\Gamma_0(2); R')$ by examining the spectral sequence associated with the sequence

$$1 \to \overline{\Gamma(2)} \to \overline{\Gamma_0(2)} \to \mathbb{Z}/2\langle \gamma_\infty \rangle \to 1. \tag{8}$$

The following result makes the spectral sequence computable.

Lemma 22 (Nishida [8]). *$\overline{\Gamma(2)}$ is a free group with generators γ_0^2 and γ_∞^2.*

Lemma 23.

$$H^1(\Gamma_0(2); R') \cong (H^1(\overline{\Gamma(2)}; R'))^{\gamma_\infty},$$

where the action of γ_∞ comes from the short exact sequence (8).

Proof. Consider the spectral sequence associated with the short exact sequence:

$$E_2^{r,s} = H^r(\mathbb{Z}/2\langle \gamma_\infty \rangle; H^s(\overline{\Gamma(2)}; R')).$$

Since $\overline{\Gamma(2)}$ is a free group, $H^s(\overline{\Gamma(2)}; R') = 0$ for $s > 1$. Further, $\overline{\Gamma(2)}$ acts trivially on R', so $H^0(\overline{\Gamma(2)}; R') = R'$. For $r > 0$ we have

$$E_2^{r,0} = H^r(\mathbb{Z}/2\langle \gamma_\infty \rangle; R') = 0$$

since $\frac{1}{2} \in R'$. Thus, $E_2^{r,s} = 0$ unless $s = 1$, or $r = 0$ and $s = 0$. Therefore, there are no differentials and $E_2^{0,1} = H^1(\overline{\Gamma_0(2)}; R')$. Thus,

$$\begin{aligned} H^1(\Gamma_0(2); R') &\cong H^1(\overline{\Gamma_0(2)}; R') \\ &\cong E_2^{0,1} \\ &= H^0(\mathbb{Z}/2\langle \gamma_\infty \rangle; H^1(\overline{\Gamma(2)}; R')) \\ &\cong (H^1(\overline{\Gamma(2)}; R'))^{\gamma_\infty}. \qquad \square \end{aligned}$$

To determine the action of γ_∞, we will use the following general properties of free groups.

Lemma 24. *Let F be a free group with generators $a_1, a_2, \ldots, a_r$. Then for any F module M,*

$$H^1(F; M) \cong (\oplus_{i=1}^r M)/I,$$

where I is the submodule generated by elements of the form

$$((a_1 - 1)\cdot m, (a_2 - 1)\cdot m, \ldots, (a_r - 1)\cdot m)$$

with $m \in M$.

Proof. Consider the free $\mathbb{Z}F$ resolution

$$0 \longrightarrow I_F \longrightarrow \mathbb{Z}F \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where ε is the augmentation map and I_F is the augmentation ideal of $\mathbb{Z}F$. Then I_F is a free $\mathbb{Z}F$ module with generators $a_1 - 1, a_2 - 1, \ldots, a_r - 1$. Thus,

$$H^1(F; M) \cong \frac{Hom_{\mathbb{Z}F}(I_F, M)}{Im(Hom_{\mathbb{Z}F}(\mathbb{Z}F, M))}.$$

But $Hom(I_F, M) \cong \oplus_{i=1}^r M$ where

$$f \leftrightarrow (f(a_1 - 1), f(a_2 - 1), \ldots, f(a_r - 1)).$$

Let $g \in Hom_{\mathbb{Z}F}(\mathbb{Z}F, M)$. Then $Im(g)$ is determined by its value on the $a_i - 1$. But $g: \mathbb{Z}F \to M$ is a $\mathbb{Z}F$ map, so

$$g(a_i - 1) = (a_i - 1)\cdot g(1).$$

Therefore, the image of g in $Hom_{\mathbb{Z}F}(I_F, M)$ corresponds to

$$((a_1 - 1)\cdot m, (a_2 - 1)\cdot m, \ldots, (a_r - 1)\cdot m) \in \oplus_{i=1}^r M,$$

where $m = g(1)$. □

Corollary 25. *If F is a free group with r generators that acts trivially on M, then*

$$H^1(F; M) \cong \oplus_{i=1}^r M.$$

Proof. Since F acts trivially on M, $(a_i - 1)\cdot m = 0$ for all $m \in M$. The claim follows immediately from the preceding lemma. □

Lemma 26.

$$(H^1(\overline{\Gamma(2)}; R'))^{\gamma_\infty} \cong R'.$$

Proof. For this proof, let $a = \gamma_\infty^2$ and $b = \gamma_0^2$. Then we know that

$$H^1(\overline{\Gamma(2)}; R') \cong Hom_{\mathbb{Z}\overline{\Gamma(2)}}(\mathbb{Z}\overline{\Gamma(2)}, R') \cong R'\langle f\rangle \oplus R'\langle g\rangle,$$

where $f, g \in Hom_{\mathbb{Z}\overline{\Gamma(2)}}(\mathbb{Z}\overline{\Gamma(2)}, R')$ are defined by

$$f(a-1) = 1, \qquad f(b-1) = 0,$$

$$g(a-1) = 0, \qquad g(b-1) = 1.$$

First, we will find the action of γ_∞ on f.

$$\begin{aligned}(\gamma_\infty \cdot f)(a-1) &= \gamma_\infty \cdot f(\gamma_\infty^{-1}(a-1)\gamma_\infty)\\ &= \gamma_\infty \cdot f(a-1) \quad \text{since } a = \gamma_\infty^2\\ &= 1 \quad \text{since } SL_2(\mathbb{Z}) \text{ acts trivially on } R'.\end{aligned}$$

In the same way, we get

$$(\gamma_\infty \cdot f)(b-1) = \gamma_\infty \cdot f(\gamma_\infty^{-1}(b-1)\gamma_\infty) = f(\gamma_\infty^{-1} b\gamma_\infty - 1).$$

A straightforward calculation shows

$$\gamma_\infty^{-1} b\gamma_\infty = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} = -Id b^{-1}a \equiv b^{-1}a$$

since we are modding by $-Id$. Thus,

$$\begin{aligned}(\gamma_\infty \cdot f)(b-1) &= f(b^{-1}a - 1)\\ &= f(b^{-1}(a-1) - b^{-1}(b-1))\\ &= b^{-1}f(a-1) - b^{-1}f(b-1)\\ &= 1.\end{aligned}$$

Therefore, $\gamma_\infty \cdot f = f + g$.

In the same way, we obtain

$$(\gamma_\infty \cdot g)(a-1) = g(a-1) = 0$$

and

$$(\gamma_\infty \cdot g)(b-1) = g(a-1) - g(b-1) = -1.$$

Thus, $\gamma_\infty \cdot g = -g$.

Therefore, the action of γ_∞ on $R'\langle f\rangle \oplus R'\langle g\rangle$ is given by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$. The $+1$ eigenspace has basis

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2f + g.$$

Therefore,

$$(H^1(\overline{\Gamma(2)}; R'))^{\gamma_\infty} \cong R'\langle 2f + g\rangle. \qquad \square$$

Proposition 27. *$H^1(\Gamma_0(2); \mathbb{R}) = \mathbb{R}$ and $M_2^{\mathbb{R}}(\Gamma_0(2)) = \mathbb{R}$. Then*

$$\varphi : M_{2k+2}^{\mathbb{R}}(\Gamma_0(2)) \xrightarrow{\cong} H^{4k+1}(X_{\Gamma_0(2)}^-; \mathbb{R})$$

for all $k \geq 0$.

Proof. Lemmas 23 and 26 show that $H^1(\Gamma_0(2); \mathbb{R}) = \mathbb{R}$. Shimura shows in [10, Theorem 2.23, p. 46] that $M_2^{\mathbb{R}}(\Gamma_0(2)) = \mathbb{R}$. Proposition 18 shows that φ is injective, and therefore, must be an isomorphism. $\square$

5. Landweber's elliptic cohomology theories

In [7] Landweber defines a connective cohomology theory whose coefficients correspond to the ring of modular forms for $\Gamma_0(2)$. Let $R' = \mathbb{Z}(\frac{1}{2})$ and δ and ε be indeterminates of weight 4 and 8, respectively. By using the Sullivan-Baas construction and the Jacobi quartic $Y^2 = 1 - 2\delta X^2 + \varepsilon X^4$, Landweber produces a connective cohomology theory Ell^c with coefficient ring

$$Ell_*^c \cong R'[\delta, \varepsilon].$$

He shows that Ell_{4k}^c corresponds to the modular forms for $\Gamma_0(2)$ of weight $2k$ with coefficients in R'. Thus,

$$Ell_{4k}^c \cong M_{2k}^{R'}(\Gamma_0(2)). \tag{9}$$

In addition to the connective theory, Landweber also constructs a periodic theory Ell^p based on the Jacobi quartic with

$$Ell_*^p \cong R'[\delta, \varepsilon, \Delta^{-1}],$$

where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$. Rather than using the Sullivan–Baas construction, he uses the Exact Functor Theorem [6] to construct the theory.

The main advantage that the periodic theory has over the connective theory lies in the righ structure of formal group laws and the existence of the algebraic map $MU_* \to Ell_*^p$. With this map, one can construct stable operations in the periodic theory.

Let $\tilde{X}_\Gamma$ denote $\Sigma^3 X_{\Gamma_0(2)}^-$, the third suspension of $X_{\Gamma_0(2)}^-$. Notice that the cohomology of $\tilde{X}_\Gamma$ with coefficients in R is isolated in dimensions $4k$ since the cohomology of X_Γ is non-zero only in dimensions $4k + 1$ by Corollary 6.

$$\begin{tikzcd} \tilde{X}_\Gamma & ell^c \\ S^4 \arrow[u, "i"] \arrow[r, "f=Id"] & S^4 \arrow[u, "i"'] \end{tikzcd}$$

Fig. 4. The map to lift.

We have an algebraic relationship between $\tilde{X}_\Gamma$ and Ell^c given by

$$\begin{aligned} Ell^c_{4k+4} \otimes \mathbb{R} &\cong M^{\mathbb{R}}_{2k+2}(\Gamma_0(2)) \quad \text{by (9)} \\ &\cong H^{4k+1}(X^-_{\Gamma_0(2)}; \mathbb{R}) \quad \text{by Proposition 27} \\ &\cong H^{4k+4}(\tilde{X}_\Gamma; \mathbb{R}). \end{aligned}$$

Proposition 28. *Let ell^c denote the 3 connected cover of Ell^c. There exists a non-trivial geometric map $\widetilde{X_\Gamma} \to ell^c$ that is an equivalence through dimension 4.*

Proof. Notice that the first cell in $\widetilde{X_\Gamma}$ occurs in dimension 4 and that there is a single cell in this dimension since $H^4(\widetilde{X_\Gamma}; \mathbb{R}) = H^1(\Gamma_0(2); \mathbb{R}) = \mathbb{R}$ by Corollary 6 and Proposition 27. Since ell^c is the 3 connected cover, the first cell in ell^c also occurs in dimension 4. Thus, we have an equivalence f on the bottom cells, and we need to show that we can lift the map to all of $\tilde{X}_\Gamma$ as in Fig. 4.

Following the standard obstruction theory arguments, the obstruction to lifting f gives an element in $H^5(\tilde{X}_\Gamma; \Pi_4 ell^c)$. By the Universal Coefficient Theorem,

$$\begin{aligned} H^5(\tilde{X}_\Gamma; ell^c_4) &\cong (H^5(\tilde{X}_\Gamma; \mathbb{R}) \otimes ell^c_4) \oplus Tor_1(H^6(\tilde{X}_\Gamma; R), ell^c_4) \\ &= (0 \otimes ell^c_4) \oplus Tor_1(0, ell^c_4), \end{aligned}$$

where the cohomology groups are 0 by Corollary 6. Thus, there are no obstructions, and the map extends to the 5-skeleton of $\tilde{X}_\Gamma$.

In general, the obstructions to lifting f to all of $\tilde{X}_\Gamma$ arise in the cohomology groups $H^{j+1}(\tilde{X}_\Gamma; ell^c_j)$. Since $ell^c_j = 0$ unless j is divisible by 4, we only need worry about the groups $H^{4k+1}(\tilde{X}_\Gamma; ell^c_{4k})$. But then the same argument using the Universal Coefficient Theorem holds, giving that $H^{k+1}(\tilde{X}_\Gamma; ell^c_{4k}) = 0$. Thus, there are no obstructions, and the map lifts to all of $\tilde{X}_\Gamma$. □

We will now construct a stable splitting of X_Γ and X_{Γ^-}. Let Γ be a congruence subgroup, and if $\Gamma = \Gamma_0(2)$, assume $p \geq 3$; otherwise, assume $p \geq 5$.

Let $\mathbb{C}P^\infty_{\hat{p}}$ denote the completion at p of infinite complex projective space. Define $Y_\Gamma = E\Gamma \times_\Gamma (\mathbb{C}P^\infty_{\hat{p}})^{\times 2}$. Then X_Γ and Y_Γ have the same mod p cohomology since we have a commutative diagram of fibrations as in Fig. 5 where the outside maps are isomorphisms in mod p cohomology.

Our goal is to construct a stable splitting of Y_Γ by finding primitive orthogonal idempotents that act on Y_Γ. Let $\mathbb{Z}/p^\times$ denote the multiplicative units in $\mathbb{Z}/p$.

$$\begin{tikzcd}
(\mathbb{C}P^{\infty})^{\times 2} \arrow[r] \arrow[d] & X_\Gamma \arrow[r] \arrow[d] & B\Gamma \arrow[d] \\
(\mathbb{C}P_{\hat{p}}^{\infty})^{\times 2} \arrow[r] & Y_\Gamma \arrow[r] & B\Gamma
\end{tikzcd}$$

Fig. 5. $H^*(X_\Gamma; \mathbb{Z}/p) \cong H^*(Y_\Gamma; \mathbb{Z}/p)$.

Then $\mathbb{Z}/p^\times$ is a cyclic group of order $p-1$ and $\mathbb{Z}/p^\times \hookrightarrow \mathbb{Z}_{\hat{p}}$. Let

$$D = \left\{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \middle| d \in \mathbb{Z}/p^\times \right\} \subseteq GL_2(\mathbb{Z}_{\hat{p}}).$$

Since $\mathbb{Z} \hookrightarrow \mathbb{Z}_{\hat{p}}$ is a ring map, we have $SL_2(\mathbb{Z}) \hookrightarrow GL_2(\mathbb{Z}_{\hat{p}})$. Thus, we can consider Γ as a subgroup of $GL_2(\mathbb{Z}_{\hat{p}})$. Notice that D is in the center of $GL_2(\mathbb{Z}_{\hat{p}})$ and acts trivially on Γ by conjugation. As before, we will let D act on $(\mathbb{C}P_{\hat{p}}^{\infty})^{\times 2}$ by right multiplication. Then D acts on Y_Γ by the following.

Proposition 29. *If $\tau \in GL_2(\mathbb{Z}_{\hat{p}})$ acts on Γ by conjugation, then τ acts on Y_Γ.*

Proof. The proof is identical to the proof of Proposition 12. □

Let $\mathbb{F}_p$ denote the finite field of order p. Then we will construct the idempotents in the group ring $\mathbb{F}_p[D]$ from a set of standard idempotents $e_0, e_1, \ldots, e_{p-2} \in \mathbb{F}_p[\mathbb{Z}/p^\times]$. First, we recall the definition of the e_i from [1] and how they split $\mathbb{C}P_{\hat{p}}^{\infty}$.

Let d be a generator of the cyclic group $\mathbb{Z}/p^\times$ that represents multiplication by $a \in \mathbb{Z}/p$. By [1] we have primitive orthogonal idempotents defined by

$$e_i = \prod_{\substack{j=0 \\ j \neq i}}^{p-2} \frac{d - a^j}{a^i - a^j}.$$

The action of d on $H^*(\mathbb{C}P_{\hat{p}}^{\infty}; \mathbb{Z}/p) \cong \mathbb{Z}/p[x]$ is given by

$$d \cdot x = ax \;\Rightarrow\; d \cdot x^i = (d \cdot x)^i = a^i x^i.$$

However, $a \in \mathbb{Z}/p$ acts by multiplication by a: $a \cdot x^i = ax^i$.

It is well known that these idempotents split $\mathbb{C}P_{\hat{p}}^{\infty}$ dimensionwise.

Proposition 30 *Stably, $\mathbb{C}P_{\hat{p}}^{\infty}$ splits into $p-1$ pieces:*

$$\mathbb{C}P_{\hat{p}}^{\infty} \cong \bigvee_{i=0}^{p-2} e_i \cdot \mathbb{C}P_{\hat{p}}^{\infty},$$

where $e_i \cdot \mathbb{C}P_{\hat{p}}^{\infty}$ has non-zero mod p cohomology in dimensions $2k$ with $k \equiv i \bmod p-2$.

We can construct idempotents $f_i \in \mathbb{F}_p[D]$ similar to the e_i by

$$f_i = \prod_{\substack{j=0 \\ j \neq i}}^{p-2} \frac{\delta - a^j}{a^i - a^j},$$

where $\delta = \left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)$. Then the action of f_i on $H^*((\mathbb{C}P_{\hat{p}}^{\infty})^{\times 2}; \mathbb{Z}/p) \cong \mathbb{Z}/p[x, y]$ is very similar to the action of e_i on $\mathbb{Z}/p[x]$ since

$$\delta \cdot x = d \cdot x = ax, \quad \text{and} \quad \delta \cdot y = d \cdot y = ay$$

giving

$$\delta \cdot x^{k-j} y^j = (\delta \cdot x)^{k-j} (\delta \cdot y)^j = a^k x^{k-j} y^j.$$

Thus, in determining the action of δ on a homogeneous polynomial, all that matters is the degree of the polynomial. And f_i acts on $\mathbb{Z}/p[x, y]^{2k}$ in the same way that e_i acts on $\mathbb{Z}/p[x]^{2k}$.

Proposition 31. *The idempotents $f_0, f_1, \ldots, f_{p-2}$ split Y_Γ dimensionally. In particular, $f_i \cdot Y_\Gamma$ has non-zero cohomology in dimensions $2k + 1$ with $k \equiv i \bmod p - 1$.*

Proof. First notice that $H^{2k+1}(Y_\Gamma; \mathbb{Z}/p) \cong H^1(\Gamma; H^{2k}((\mathbb{C}P_{\hat{p}}^{\infty})^{\times 2}; \mathbb{Z}/p))$ as in Proposition 4. Let $f: \Gamma \to \mathbb{Z}/p[x, y]^{2k}$ be a derivation. Then for any $\gamma \in \Gamma$,

$$\begin{aligned} (\delta \cdot f)(\gamma) &= \delta \cdot f(\delta^{-1} \gamma \delta) \\ &= \delta \cdot f(\gamma) \end{aligned}$$

since δ commutes with Γ. Thus, the action of δ on Y_Γ is completely determined by its action on $(\mathbb{C}P_{\hat{p}}^{\infty})^{\times 2}$.

As in Proposition 30 above,

$$f_i \cdot x^{k-j} y^j = \begin{cases} x^{k-j} y^j & \text{if } k \equiv i \bmod p - 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Recall that Theorem 19 did not deal with the full spectrum X_Γ, but with a summand X_{Γ^-}. If $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts on Γ by conjugation, then Proposition 29 shows that σ acts on Y_Γ. Therefore, we have a splitting analogous to that for X_Γ:

$$Y_\Gamma \simeq Y_{\Gamma^+} \vee Y_{\Gamma^-}.$$

Proposition 32. *D acts on Y_{Γ^-}. Thus, the idempotents f_i act on Y_{Γ^-}, and $f_i \cdot Y_{\Gamma^-}$ has non-zero cohomology in dimensions $2k + 1$ with $k \equiv i \bmod p - 1$.*

Proof. Since D commutes with σ, δ commutes with the idempotents defined in Corollary 13 used to split X_Γ and Y_Γ. Thus, the f_i act on Y_{Γ^-} and detect the same dimensions as in Proposition 31. $\square$

Unlike D, the action of σ on the cohomology of Y_Γ is not completely determined by the dimension. That is,

$$\sigma \cdot x^{k-j} y^j = x^{k-j}(-y)^j$$

and σ acts non-trivially on Γ by conjugation. If $f: \Gamma \to \mathbb{Z}/p[x, y]^{2k}$ is a derivation representing a class in $H^{2k+1}(Y_\Gamma; \mathbb{Z}/p)$, then $\sigma \cdot f$ depends on more than k since

$$(\sigma \cdot f)(\gamma) = \sigma \cdot f(\sigma \gamma \sigma).$$

Thus, the splitting we get from the idempotents formed from σ is not dimensionwise.

6. A splitting of Baker's periodic theory

Baker [2] also constructs a periodic theory by the Exact Functor Theorem, but he begins with the Weierstrauss cubic $Y^2 = 4X^3 - g_2 X - g_3$. Then the cubic defines an elliptic curve over $R[g_2, g_3]$ where $R = \mathbb{Z}(\frac{1}{6})$. For the Exact Functor Theorem to apply, we need to invert the discriminant

$$\Delta_{Ell} = g_2^3 - 27 g_3^2.$$

Then we obtain a cohomology theory $\widetilde{Ell}$ with

$$\widetilde{Ell}_* \cong R[g_2, g_3, \Delta_{Ell}^{-1}].$$

Baker then associates $\widetilde{Ell}_*$ with the ring of modular functions for $SL_2(\mathbb{Z})$ that are holomorphic on $\mathscr{H}$ with a possible pole at ∞. As with Landweber's construction, the grading is cut into half: $\widetilde{Ell}_{4k}$ corresponds to the functions of weight $2k$.

We can use the elliptic cohomology Adams operations defined by Baker [2] to construct idempotents $\tilde{f}_i \in \mathbb{F}_p[\mathbb{Z}/p^\times]$ that split Baker's theory $\widetilde{Ell}$ at a prime $p > 3$. Let $a > 3$ be an integer, and let Ell^a denote the cohomology theory

$$Ell^{a*}(-) = \widetilde{Ell}^*(-) \otimes \mathbb{Z}\left(\frac{1}{a}\right).$$

Notice that if we localize at a prime p that is relatively prime to a, then Ell^a and $\widetilde{Ell}$ define the same theory. If q is a prime factor of a, then [2] gives a natural multiplicative transformation

$$\psi^q : \widetilde{Ell}^*(-) \to Ell^{a*}(-)$$

which is characterized by its behaviour on the coefficient rings:

$$\psi^q(\eta) = q^k \eta \quad \text{if } \eta \in \widetilde{Ell}_{2k}.$$

Let $p > 3$ be prime, and suppose that we have localized $\widetilde{Ell}$ at p. Choose a so that multiplication by a generates the cyclic group $\mathbb{Z}/p^\times$. Then a and p are relatively prime, and the theories $\widetilde{Ell}$ and Ell^a are the same. Thus, for all prime factors q of a, we have the operations

$$\psi^q : \widetilde{Ell}^*(-) \to \widetilde{Ell}^*(-).$$

We can compose these operations to produce an operation

$$\psi^a : \tilde{E}ll^*(-) \to \tilde{E}ll^*(-)$$

such that

$$\psi^a(\eta) = a^k \eta \quad \text{if } \eta \in \tilde{E}ll_{2k}.$$

Thus, the action of ψ^a on $\tilde{E}ll_*$ is completely determined by the dimension of η. This situation is analogous to that where we constructed the idempotents f_i. We define the idempotents $\tilde{f}_i \in \mathbb{F}_p[\mathbb{Z}/p^\times]$ by

$$\tilde{f}_i = \prod_{\substack{j=0 \\ j \neq i}}^{p-2} \frac{\psi^a - a^j}{a^i - a^j}.$$

Following the same argument as in Proposition 30, we have the following result.

Proposition 33. *Localized at $p > 3$, Baker's theory $\tilde{E}ll$ splits into $p - 1$ pieces:*

$$\tilde{E}ll \cong \bigvee_{i=0}^{p-2} \tilde{f}_i \cdot \tilde{E}ll,$$

where $\tilde{f}_i \cdot \tilde{E}ll$ has non-zero homotopy in dimensions $2k$ with $k \equiv i \bmod p - 2$.

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